Alternating groups as monodromy groups in positive characteristic

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Abstract

Let X be a generic curve of genus g defined over an algebraically closed field k of characteristic $p \geq 0$. We show that for n sufficiently large there exists a tame rational map $f : X \to \mathbb{P}^1_k$ of degree n with monodromy group A_n. This generalizes a result of Magaard–Völklein to positive characteristic.

1 Introduction

1.1 Let k be an algebraically closed field of characteristic $p \geq 0$. We denote by k_0 the prime field of k (i.e. $k_0 = \mathbb{Q}$ if $p = 0$, $k = \mathbb{F}_p$ if $p > 0$). For $g \geq 0$, we write $\mathcal{M}_g$ for the (coarse) moduli space of curves of genus g in characteristic p. This is a smooth, quasi-projective and geometrically irreducible variety over k_0, see [3]. Its dimension is $3g-3$ (resp. g) for $g \geq 2$ (resp. for $g = 0, 1$).

Let X be a smooth projective curve of genus g over k. It corresponds to a k-rational point $x : \operatorname{Spec} k \to \mathcal{M}_g$. We say that X is *generic* if the image of x is Zariski dense in $\mathcal{M}_g$.

Let $f : X \to \mathbb{P}^1_k$ be a non-constant rational function on X. We say that f is *tame* if the extension of function fields $k(X)/k(f)$ is separable and at most tamely ramified. The degree of the extension $k(X)/k(f)$ is called the *degree* of f. The Galois group of the Galois closure of $k(X)/k(f)$ is called the *monodromy group* of f. The aim of this note is to prove the following theorem.

Theorem 1 *Let $g \geq 0$ and $n \geq 3$. Let X be a generic curve of genus g, defined over an algebraically closed field k of characteristic $p \geq 0$. Then the curve X admits a tame rational function $f : X \to \mathbb{P}^1_k$ of degree n with alternating monodromy group, in each of the following cases:*

(a) If $p \neq 2, 3$ and $n \geq \max(g+3, 2g+1)$.

(b) If $p = 2$, $n \geq \max(g+3, 2g+1)$ and $n+g$ is odd.

(c) If $p = 2$, $n \geq \max(g+6, 2g+3)$ and $n+g$ is even.

(d) If $p = 3$ and $n \geq \max(7, g+6, 2g+1)$.

In characteristic $p = 0$, this has recently been proved by Magaard and Völklein [12] (except for the case $(g,n) = (2,5)$). The cases $g = 1$ and $p = 0$ had been proved earlier by Fried, Klassen and Kopeliovich [4]. Also in characteristic 0, Artebani and Pirola [1] have shown that *every* curve admits a tame rational function of degree n with alternating monodromy group, provided that $n \geq 12g + 4$. In characteristic $p \neq 3$, Schröer [14] has proved that for every $g \geq 0$ there exists *some* curve of genus g admitting a rational function with monodromy group A_n, for certain values of n.

Note that there are classical analogs of these results for the symmetric group. In fact, one knows that every curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p \neq 2$ admits a tame rational function of degree n with monodromy group S_n, provided that $2n - 2 \geq g$ ([6]). However, in characteristic 2, it seems to be unknown whether every curve admits a tame rational function at all – no matter what the degree and the monodromy group are.

Our results are somewhat more precise than Theorem 1 above: the tame cover $f : X \to \mathbb{P}^1_k$ can be chosen in such a way that its inertia groups are all generated by 3-cycles (except if $p = 3$ or if $p = 2$ and $n + g$ is odd). If $p \neq 2, 5$ and $n \geq \max(g + 6, 2g + 1)$, the cover f can be chosen in such a way that its inertia groups are all generated by double transpositions (with the possible exception $(p, g, n) = (3, 0, 6)$). See the statement of Theorem 10.

For $p \neq 2, 3$ our results are optimal, in the following sense. If $n < g + 3$ or $n < 2g + 1$ then the generic curve of genus g does *not* admit a tame rational function of degree n with alternating monodromy group. (For $g \geq 3$, this is shown in [12], see the proof of Theorem 3.3 in *loc.cit.* For $g < 3$, one can use similar arguments.)

For $p = 2$ or $p = 3$, it is unlikely that our results are optimal. For instance, we believe (but were not able to show) that the generic curve of genus 1 in characteristic 2 admits a tame rational function of degree 5 with alternating monodromy. If this were the case, then the situation in characteristic 2 would be the same as in characteristic 0, i.e. the condition '$n + g$ odd' in Theorem 1 (b) would be unnecessary (except for the case $(g,n) = (0,4)$, which does *not* occur in characteristic 2). This would then also be an optimal result. But at the moment, there are infinitely many pairs (g,n) which, to our knowledge, may or may not occur in characteristic 2. In characteristic 3, there are only finitely many such cases. See Section 3.3 for a list of all open cases.

1.2 In order to show the existence of rational functions with alternating monodromy, the authors of [12] use *Hurwitz spaces*, i.e. moduli space for covers of $\mathbb{P}^1$. In fact, the main result of [12] is stated in the following form: the natural map $\mathcal{H}_{r,n} \to \mathcal{M}_g$ from a certain Hurwitz space to the moduli space of curves (which maps the isomorphism class of a cover $f : X \to \mathbb{P}^1$ to the isomorphism class of the curve X) has a dense image. In [5], Hurwitz spaces are constructed in characteristic 0, using tools from topology and the theory of complex analytic functions. From this point of view, it seems difficult to extend the results of [12] to positive characteristic. However, it is shown in [6] and [16] that Hurwitz

spaces can also be constructed in a purely algebraic way and therefore make sense in positive characteristic, too. Moreover, for almost all primes p a given Hurwitz space has good reduction from characteristic 0 to characteristic p (in the case of the Hurwitz space $\mathcal{H}_{r,n}$ used in [12], this is true for all primes $p > n$). Using this good reduction result, it is then easy to extend the results of [12] to characteristic p, provided that $p > n$.

However, for small primes p this kind of argument does not work. For instance, if $p \leq n$ the Hurwitz space $\mathcal{H}_{n,r}$ may have bad reduction to characteristic p. This makes it very difficult to study $\mathcal{H}_{n,r} \otimes \mathbb{F}_p$, using results on $\mathcal{H}_{n,r} \otimes \mathbb{Q}$. The same phenomenon prevented Fulton [6] from proving the irreducibility of $\mathcal{M}_g \otimes \mathbb{F}_p$ for $p \leq 2g+1$.

Luckily, it turns out that most of the arguments in [12] (for instance, the induction step from (g,n) to $(g+1,n+1)$) can be carried out in a purely algebraic way. In fact, Hurwitz spaces are not really essential; the key results that are used are the geometry of the boundary of $\mathcal{M}_g$ and the deformation theory of tame covers, developed in [7]. Both these tools are algebraic and work in any characteristic. The situation is different for the proof of Theorem 1 for small values of g and n, where the induction process starts. Here the arguments in [12] are based on Riemann's Existence Theorem and do not carry over to characteristic p if p is small. This is the reason why the statement of Theorem 1 is weaker for $p = 2, 3$.

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2 The two induction steps

2.1 Deformation of admissible covers Let k be an algebraically closed field. We choose a compatible system (ζ_n) of nth roots of unity in k, where n runs over all natural numbers prime to the characteristic of k. Let t denote a transcendental element over k. Our goal is to construct tame covers $f_\eta : X_\eta \to \mathbb{P}^1_{k((t))}$ over the field $k((t))$ by deforming a given tame cover $f_s : X_s \to Z_s$ between singular curves. The standard general reference for the deformation theory of tame covers is of course [7]. For the particular results that we use, we refer to [13], [8] or [17].

Let G be a finite group. For $i = 1, 2$, we have a subgroup $G_i \subset G$, a tame G_i-Galois cover $h_i : Y_i \to \mathbb{P}^1_k$ between smooth projective curves over k, and a closed point $y_i \in Y_i$. We assume that the datum (G, G_i, h_i, y_i) has the following properties. First, we assume that G is generated by its subgroups G_1 and G_2. For the second condition, let $g_i \in G_i$ be the canonical generator of the stabilizer of y_i, with respect to (ζ_n). (An element $g \in G_i$ with $g(y_i) = y_i$ is called a *canonical generator* if there exists a formal parameter u at y_i such that $g^* u = \zeta_n \cdot u$, with $n = \mathrm{ord}(g)$.) Then we demand that $g_1 = g_2^{-1}$. We denote by n_0 the order of $g_1 = g_2^{-1}$.

Given (G, G_i, h_i, y_i) satisfying the above conditions, we construct a tame G-Galois cover $h_s : Y_s \to Z_s$ between semistable curves over k, as follows. We set

$$Y_s := \Big(\mathrm{Ind}_{G_1}^G(Y_1) \coprod \mathrm{Ind}_{G_2}^G(Y_2)\Big)/\sim,$$

where $\sim$ denotes the following equivalence relation. A point $y \in \mathrm{Ind}_{G_1}^G(Y_1)$ is equivalent to a point $y' \in \mathrm{Ind}_{G_2}^G(Y_2)$ if and only there exists an element $g \in G$ with $y = g(y_1)$ and $y' = g(y_2)$. It is easy to see that the set Y_s is naturally equipped with the structure of a semistable curve over k and with a k-linear action of G. The curves $Y_1 \subset Y_s$ and $Y_2 \subset Y_s$ are irreducible components of Y_s, with stabilizer G_1 and G_2, respectively. Moreover, the points $y_1 \in Y_1$ and $y_2 \in Y_2$ correspond to the same (singular) point of Y_s. We define $Z_s := Y_s/G$ as the quotient scheme. The scheme Z_s is a semistable curve over k, with two irreducible components Z_1, Z_2 which meet in one points. The components Z_i can be identified with $\mathbb{P}^1_k$, via the covers h_i. The natural map $h_s : Y_s \to Z_s$ is a *tame admissible cover* ([17]).

It is easy to construct a semistable curve $\mathcal{Z}$ over $\mathrm{Spec}\, k[[t]]$ with special fiber Z_s and with generic fiber $Z_\eta = \mathbb{P}^1_{k((t))}$, satisfying the following additional property: the complete local ring of $\mathcal{Z}$ at the singular point of the special fiber is isomorphic to $k[[t, u, v \mid uv = t^{n_0}]]$. Let $\bar{\tau}_1, \ldots, \bar{\tau}_s \in Z_1 = \mathbb{P}^1_k$ denote the branch points of h_1 distinct from $h_1(y_1)$, and $\bar{\tau}_{s+1}, \ldots, \bar{\tau}_r$ the branch points of h_2 distinct from $h_2(y_2)$. Lift these points to $k[[t]]$-rational points $\tau_1, \ldots \tau_r$ of $\mathcal{Z}$. By [17], Theorem 3.1.1, there exists a tame G-Galois cover $h : \mathcal{Y} \to \mathcal{Z}$ between semistable curves over $k[[t]]$, étale over $\mathcal{Z} - \{\tau_1, \ldots, \tau_r\}$ whose special fiber is equal to the cover h_s. The generic fiber $h_\eta : Y_\eta \to \mathbb{P}^1_{k((t))}$ is a tame G-Galois cover between smooth projective curves, with r branch points. We say that h_η is a *smooth Galois cover* associated to (G, G_i, h_i, y_i).

Suppose that $G \subset S_n$ is a transitive permutation group on n letters, and let $H \subset G$ denote the stabilizer of 1 in G. Let $\mathcal{X} := \mathcal{Y}/H$ denote the quotient scheme. Then $\mathcal{X}$ is a semistable curve over $k[[t]]$, with generic fiber $X_\eta = Y_\eta/H$ and special fiber $X_s = Y_s/H$ (for the second equality we have used the fact that the cover $h_s : Y_s \to Z_s$ is separable). Moreover, h_η is the Galois closure of the cover $f_\eta : \mathcal{X}_\eta \to \mathbb{P}^1_{k((t))}$. Therefore, the cover f_η is tame, with monodromy group G. We say that f_η is a *smooth cover* associated to $(G \subset S_n, G_i, h_i, y_i)$.

It is easy to describe the special fiber X_s and the admissible cover $f_s : X_s \to Z_s$ in terms of the datum $(G \subset S_n, G_i, h_i, y_i)$. For instance, the irreducible components of X_s lying above the component $Z_i \subset Z_s$ are in bijection with the orbits of the G_i-action on $\{1, \ldots, n\}$. For each orbit $O \subset \{1, \ldots, n\}$, the restriction of f_s to the component $X_O \subset X_s$ is isomorphic to the quotient cover of $h_i : Y_i \to Z_i = \mathbb{P}^1_k$ corresponding to the stabilizer of some element of O.

2.2 A useful lemma Let X be a generic curve of genus $g \geq 1$, defined over an algebraically closed field k of characteristic $p \geq 0$. Let $f : X \to \mathbb{P}^1_k$ be a tame cover, with monodromy group G and with branch points $t_1, \ldots, t_r$, $r \geq 3$. Without loss of generality, we may assume that $t_1 = 0$, $t_2 = 1$ and $t_3 = \infty$. We

say that f has *generic branch points* if the branch points $t_4, \ldots, t_r$, considered as elements of k, are algebraically independent over the prime field $k_0 \subset k$.

The following lemma will be useful in the proof of Theorem 1.

Lemma 2 *Let k'/k be a field extension, $z : \mathrm{Spec}\, k' \to \mathbb{P}^1_k$ a generic point and $x_1, x_2 : \mathrm{Spec}\, k' \to X$ two distinct k'-rational points with $z = f(x_1) = f(x_2)$. Suppose that one of the following conditions holds.*

(a) We have $g = 1$ and the monodromy group G is doubly transitive.

(b) We have $g \geq 2$ and f has $r > 3g$ generic branch points.

Then $(X \otimes k'; x_1, x_2)$ is a generic two-pointed curve, i.e. its classifying map $\mathrm{Spec}\, k' \to \mathcal{M}_{g,2}$ has a Zariski dense image.

Proof: Assume that Condition (a) of the Lemma holds. Let $\varphi_1 : \mathrm{Spec}\, k' \to \mathcal{M}_{1,1}$ (resp. $\varphi_2 : \mathrm{Spec}\, k' \to \mathcal{M}_{1,2}$) denote the classifying map of the one-pointed curve $(X \otimes k', x_1)$ (resp. of the two-pointed curve $(X \otimes k'; x_1, x_2)$). Recall that $\mathcal{M}_{1,1} \cong \mathbb{A}^1$ and that φ_1 is simply the j-invariant of the elliptic curve $(X \otimes k', x_1)$. It is well known that the j-invariant of an elliptic curve depends only on the underlying curve and not on the distinguished point (i.e. $\mathcal{M}_{1,1} \cong \mathcal{M}_1$). Hence we may regard φ_1 as an element of k. Since X is generic, φ_1 is transcendental over k_0. It follows that the image of φ_2 is contained in the fiber $\mathcal{M}_{1,2} \otimes k_0(\varphi_1)$ of the natural map $\mathcal{M}_{1,2} \to \mathcal{M}_{1,1}$ over $\mathrm{Spec}\, k_0(\varphi_1)$. To prove the lemma, it suffices to show that the image of φ_2 is Zariski dense in $\mathcal{M}_{1,2} \otimes k_0(\varphi_1)$. Since $\mathcal{M}_{1,2} \otimes k_0(\varphi_1)$ is 1-dimensional, it even suffices to find a k-rational place v of k' such that the image of v under φ_2 lies on the boundary of $\mathcal{M}_{1,2} \otimes k_0(\varphi_1)$. In other words, we require $x_1(v) = x_2(v)$ (here $x_i(v)$ denotes the k-rational point on X obtained by 'specializing' x_i at v).

Let $S \subset X \times_k X$ denote the locus of pairs (x, y) with $f(x) = f(y)$. It is a 1-dimensional closed subset, and the natural map $S \to \mathbb{P}^1_k$ is finite. The pair (x_1, x_2) is a k'-rational point on S which maps to the generic point of $\mathbb{P}^1_k$. Therefore, the closure of the image of (x_1, x_2) is an irreducible component S' of S, distinct from the diagonal $\Delta \subset X \times_k X$. We have to show that S' has nonempty intersection with Δ. The assumption that G is doubly transitive implies that $S - \Delta$ is irreducible. Hence we have $S' = \overline{S - \Delta}$. On the other hand, since the cover $f : X \to \mathbb{P}^1_k$ is not étale, Δ has nonempty intersection with $S' = \overline{S - \Delta}$. This proves Lemma 2, assuming Condition (a).

Assume now that Condition (b) holds. Let $k_1 \subset k$ denote the minimal algebraically closed subfield of k over which the curve X can be defined. Since X is generic, k_1 is isomorphic to the algebraic closure of the function field of $\mathcal{M}_g$. In particular, $\mathrm{tr.deg}(k_1/k_0) = 3g - 3$. Write $X = X_1 \otimes_{k_1} k$.

By assumption, the branch points $t_4, \ldots, t_r$ of f, considered as elements of k, are algebraically independent over k_0. Moreover, we have $r - 3 > 3g - 3 = \mathrm{tr.deg}(k_1/k_0)$. It follows that the cover $f : X_1 \otimes k \to \mathbb{P}^1_k$ is not isotrivial, with respect to the extension k/k_1. More precisely, the subfield $k(f) \subset k(X)$ cannot be generated over k by an element $f_1 \in k_1(X_1)$.

Let $f_V : X_1 \times V \to \mathbb{P}^1_V$ be a model of f over V, where V is a variety over k_1 (this means that f is the pullback of f_V via a generic point $\mathrm{Spec}\, k \to V$). For each closed point $v \in V$ we obtain, by specializing f_V, a tame cover $f_v : X_1 \to \mathbb{P}^1_{k_1}$. Since f is not isotrivial, there exist infinitely many points $v \in V$ which give rise to pairwise weakly non-isomorphic covers f_v. Now [12], Lemma 2.3, shows that the locus $S \subset X_1 \times_{k_1} X_1$ of pairs of points (x, y) which satisfy $f_v(x) = f_v(y)$ for all $v \in V$ is Zariski dense in $X_1 \times_{k_1} X_1$. On the other hand, since $S \to \mathbb{P}^1_{k_1}$ is finite and $f(x_1) = f(x_2) = z$ is generic, the pair (x_1, x_2) is a generic point of S. This proves Lemma 2, assuming Condition (b). □

Remark 3 Lemma 2 is very similar to [12], Lemma 2.4, at least in the case $g \geq 2$. In the case $g = 1$, Lemma 2 improves [12], Lemma 2.4, by removing the assumption that f depends on more parameters than X. This is used in the proof of Theorem 1 for the case $(g, n) = (2, 5)$.

2.3 The first induction step As before, we let X be a generic curve of genus $g \geq 0$, defined over an algebraically closed field k of characteristic $p \geq 0$. The following proposition will serve as an induction step in the proof of Theorem 1, from (g, n) to $(g, n + 2)$.

Proposition 4 *Assume that there exists a tame cover $f : X \to \mathbb{P}^1_k$ of degree $n \geq 3$, with monodromy group A_n and with r branch points. Let $k' := k((t))^{\mathrm{alg}}$ and $X' := X \otimes_k k'$. Then there exists a tame cover $f' : X' \to \mathbb{P}^1_{k'}$ of degree $n + 2$, with monodromy group A_{n+2} and with $r + 2$ branch points.*

Proof: Let $h_1 : Y_1 \to \mathbb{P}^1_k$ be the Galois closure of f. By assumption, the Galois group of h_1 is $G_1 := A_n$, which we view as a subgroup of $G := A_{n+2}$, in the obvious way. Let $y_1 \in Y_1$ be any closed point where h_1 is unramified. If $p \neq 3$ (resp. if $p = 3$), we let $h_2 : Y_2 \to \mathbb{P}^1_k$ be a cyclic cover of order 3 (resp. of order 2), ramified at two points. We identify the Galois group of h_2 with the subgroup $G_2 \subset G$ generated by the 3-cycle $(n, n+1, n+2)$ (resp. by the double transposition $(n-1, n+1)(n, n+2)$). Note that $G = \langle G_1, G_2 \rangle$ in both cases. We also choose a point $y_2 \in Y_2$ where h_2 is unramified. By the construction of §2.1, there exists a tame G-Galois cover $h' : Y' \to \mathbb{P}^1_{k'}$ which lifts the datum (G, G_i, h_i, y_i). Let $H \subset G$ be the stabilizer of 1. Set $X' := Y'/H$ and let $f' : X' \to \mathbb{P}^1_{k'}$ denote the natural map. By construction, f' is a tame cover of degree $n + 2$, with monodromy group A_{n+2} and with $r + 2$ branch points.

It remains to prove that $X' \cong X \otimes_k k'$. The Riemann–Hurwitz formula shows that the genus of X' is equal to g. Moreover, by construction, one of the components of the stable reduction of X' is isomorphic to X. It follows that X' has good reduction and specializes to a generic curve of genus g. Therefore, X' is itself generic, hence $X' \cong X \otimes_k k'$. □

2.4 The second induction step For the proof of Theorem 1, we need another induction step, going from (g, n) to $(g + 1, n + 1)$. As in the last subsection, X is a generic curve of genus $g \geq 0$ over k.

Proposition 5 *Assume that there exists a tame cover $f : X \to \mathbb{P}^1_k$ of degree $n \geq 3$, with monodromy group A_n and with r branch points. Assume, moreover, that one of the following conditions hold:*

(i) $g \leq 1$,

(ii) $g \geq 2$ *and* $r > 3g$.

Then there exists a field extension k'/k, a generic curve X' of genus $g+1$ over k' and a tame cover $f' : X' \to \mathbb{P}^1_{k'}$ of degree $n+1$, with monodromy group A_{n+1} and with $r+2$ branch points.

Proof: We are allowed to replace the field k by an arbitrary extension k'. We may therefore assume that the cover $f : X \to \mathbb{P}^1_k$ has generic branch points, see §2.2. (To see this, let $\mathcal{H}_{n,r}$ denote the Hurwitz space parameterizing tame covers of $\mathbb{P}^1$ of degree n with r branch points. By [16], Proposition 4.2.1, $\mathcal{H}_{n,r}$ is a smooth scheme of relative dimension r over $\mathbb{Z}$. The cover f of the proposition corresponds to a k-rational point on $\mathcal{H}_{n,r} \otimes k_0$. We may thus replace f by the generic cover corresponding to the connected component of $\mathcal{H}_{n,r} \otimes k_0^{\rm alg}$ on which this point lies.)

Let $z : \mathrm{Spec}\, k_1 \to \mathbb{P}^1_k$ be a generic geometric point. Let $h_1 : Y_1 \to \mathbb{P}^1_{k_1}$ denote the Galois closure of $f \otimes k_1$. We identify the Galois group of h_1 with $G_1 := A_n$, which we regard as a subgroup of $G := A_{n+1}$, in the obvious way. We choose a point $y_1 \in Y_1$ which lies above $z \in \mathbb{P}^1_{k_1}$. If $p \neq 3$ (resp. if $p = 3$), we let $h_2 : Y_2 \to \mathbb{P}^1_{k_1}$ be a cyclic cover of order 3 (resp. of order 2), ramified at two points. We identify the Galois group of h_2 with the subgroup $G_2 \subset G$ generated by the 3-cycle $(n-1, n, n+1)$ (resp. by the double transposition $(1,2)(n, n+1)$). We also choose a point $y_2 \in Y_2$ where h_2 is unramified. Let $k' := k_1((t))^{\rm alg}$. We lift the datum (G, G_i, h_i, y_i) to a tame G-Galois cover $h' : Y' \to \mathbb{P}^1_{k'}$, see §2.1. Let $H \subset G$ be the stabilizer of 1. Set $X' := Y'/H$ and let $f' : X' \to \mathbb{P}^1_{k'}$ denote the natural map. By construction, f' is a tame cover of degree $n+1$, with monodromy group A_{n+1} and with $r+2$ branch points. The Riemann–Hurwitz formula shows that X' has genus $g+1$. It remains to prove that X' is generic.

Let $\mathcal{X}$ be the semistable model of X' over $k_1[[t]]$ originating from the construction of §2.1. If $p \neq 3$, then the special fiber X_s of $\mathcal{X}$ contains the curves $X \otimes k_1$ and Y_2 as irreducible components. These components meet in two points $x_1, x_2 \in X$ with $f(x_1) = f(x_2) = z$. The other components of X_s are copies of $\mathbb{P}^1_{k_1}$. If $p = 3$, the situation is similar. The only difference is that X_s contains two copies of Y_2 as irreducible components, corresponding to the orbits of length two of $(1,2)(n, n+1)$ acting on $\{1, 2, \ldots, N\}$.

If $g = 0$, the above description of X_s shows that X' has bad reduction and is therefore a generic curve of genus 1. Hence we may assume that $g \geq 1$. Let $\mathcal{X}^{\rm stab}$ be the stable model of X' over $k_1[[t]]$ (i.e. the minimal semistable model). The curve $\mathcal{X}^{\rm stab}$ is obtained from the curve $\mathcal{X}$ by blowing down all components of X_s of genus 0 which contain less than three singular points. It follows that the special fiber of $\mathcal{X}^{\rm stab}$ is isomorphic to the curve $X \otimes k_1$, with the points x_1 and x_2 identified. Let $\varphi : \mathrm{Spec}\, k_1[[t]] \to \bar{\mathcal{M}}_{g+1}$ denote the classifying map for

$\mathcal{X}^{\mathrm{stab}}$, and write φ_η (resp. φ_s) for the restriction of φ to the generic (resp. the closed) point of Spec $k_1[[t]]$. We have to show that the image of φ_η is Zariski dense in $\mathcal{M}_{g+1}$. By our description of the special fiber, the image of φ_s lies on a boundary component T of $\bar{\mathcal{M}}_{g+1}$ which is isomorphic to $\mathcal{M}_{g,2}$. Moreover, φ_s corresponds, under this isomorphism, to the 2-pointed curve $(X \otimes k_1; x_1, x_2)$ ([11]). It follows from Lemma 2 that the image of φ_s is Zariski dense in the boundary component T (note that A_n is doubly transitive for $n \geq 4$ and that the case $n = 3$ does not occur for $g \geq 1$). But T has codimension one in $\bar{\mathcal{M}}_{g+1}$ ([3]). Therefore, the image of φ_η is Zariski dense in $\mathcal{M}_{g+1}$. This concludes the proof of Proposition 5. □

3 The main result

3.1 The case $g = 0$ Proposition 7 implies Part (a) of Theorem 1. In the proof we use the following easy lemma.

Lemma 6 *Let $p \neq 2$. Choose $\lambda \in k - \{0, 1\}$ such that*

$$\Phi(\lambda) = \sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i}^2 \lambda^i$$

is nonzero. Then there exists a cover $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree p branched at $0, 1, \lambda, \infty$ with monodromy group D_p. For $x = 0, 1, \lambda, \infty$, the inverse image of x consists of $(p-1)/2$ ramification points of order two and one unramified point.

Proof: Choose $\lambda \in k - \{0, 1\}$ such that $\Phi(\lambda)$ is nonzero. This implies that the elliptic curve E defined by $y^2 = x(x-1)(x-\lambda)$ is ordinary, see e.g. [9], Corollary IV.4.22. Let $E' \to E$ be an étale isogeny of degree p. Taking the quotient on both sides under the elliptic involution, we obtain a rational map $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree p as in the statement of the lemma. □

Proposition 7 *Let $n \geq 3$ and $p \geq 0$. Suppose that $n \neq 4$ if $p = 2$ and $n \geq 7$ if $p = 3$. Then there exists a tame cover $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree n with monodromy group A_n, defined over an algebraically closed field k of characteristic p. More precisely:*

(a) If $p \neq 2, 3$, we may choose f with $r = n - 1$ branch points and ramification type $(3, \ldots, 3)$.

(b) Suppose that one of the following holds:

(i) *$p \geq 7$ and $n \geq 6$,*

(ii) *$p = 3$ and $n \geq 7$, or*

(iii) *$p = 5$ and $n \geq 7$ is odd.*

Then we may choose f with $r = n-1$ branch points and ramification type $(2\text{-}2, \ldots, 2\text{-}2)$.

(c) *If $p = 2$ and n is odd, we may choose f with $r = n-1$ branch points and ramification type* $(3, \ldots, 3)$.

(d) *If $p = 2$ and $n \geq 6$ is even, we may choose f with $r = n-3$ branch points, and ramification type* $(5, 3\text{-}3, 3, \ldots, 3)$.

Proof: Suppose first that $p \neq 2, 3$ and consider the case of triple ramification. The case $n = 3$ is trivial, as we may take for f the function $f(x) = x^3$. For $n = 4$, we set

$$f(x) \;:=\; \frac{x^3\,(x-2)}{1-2x} \;=\; \frac{(x-1)^3\,(x+1)}{1-2x} + 1. \tag{1}$$

It is obvious that f has ramification locus $\{0, 1, \infty\}$ and ramification index 3 in each of these points. (Note that we use $p \neq 2, 3$.) The monodromy group of f is a transitive permutation group on 4 letters generated by 3-cycles, hence isomorphic to A_4. This settles the case $n = 4$. The cases $n > 4$ follow from $n = 3, 4$ by induction, using Proposition 4. This proves (a)

We now prove (b). If $p \geq 7$ and $n = 6$, we may use Riemann's Existence Theorem, since in this case p does not divide the order of A_6. For example, one checks using GAP that the tuple

$$((1,2)(3,4); (3,6)(4,5); (2,6)(3,4); (2,6)(3,5); (1,2)(4,6))$$

generates A_6.

Next we handle the case $p \neq 2$ and $n = 7$. Let $f_1 : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a separable map of degree four branched at four points $0, 1, \lambda, \infty$ with ramification type $(2, 2, 2, 2)$ and monodromy group G_1 the dihedral group of order 8. For $p \neq 2$, the existence of such a cover follows from Riemann's Existence Theorem. Write $h_1 : Y_1 \to \mathbb{P}^1_k$ for the Galois closure of f_1 and choose a point $y_1 \in Y_1$ above $x = 0 \in \mathbb{P}^1_k$. Choose an embedding of G_1 into A_7 such that G_1 acting on $\{1, 2, \ldots, 7\}$ has orbits $\{1, 2\}$, $\{3, 4, 6, 7\}$, $\{5\}$. Write (g_1, g_2, g_3, g_4) for the corresponding tuple of double transpositions. Choose $\mu \in k - \{0, 1\}$. If $p = 5$ we suppose moreover that μ is not a primitive 3rd root of unity. Let $f_2 : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a separable map of degree five branched at $0, 1, \mu, \infty$ with monodromy group the dihedral group of order 10 and with ramification type $(2\text{-}2, 2\text{-}2, 2\text{-}2, 2\text{-}2)$. If $p \neq 5$ the existence of f_2 follows from Riemann's Existence Theorem. If $p = 5$ it follows from Lemma 6. Write $h_2 : Y_2 \to \mathbb{P}^1_k$ for the Galois closure of f_2 and choose a point $y_2 \in Y_2$ above $x = 0 \in \mathbb{P}^1_k$. Let G_2 denote the Galois group of h_2. We embed G_2 into A_7 such that the action of G_2 on $\{1, 2, \ldots, 7\}$ has orbits $\{1, 2, 3, 4, 5\}$, $\{6\}$, $\{7\}$. Write (g_1', g_2', g_3', g_4') for the tuple of transpositions corresponding to the cover f_2. We may arrange things such that $g_1 = g_1'$. One checks that A_7 is generated by G_1 and G_2. Therefore the results of §2.1 imply that there exists a map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree $n = 7$ with monodromy group A_7 and ramification $(2\text{-}2, 2\text{-}2, 2\text{-}2, 2\text{-}2, 2\text{-}2, 2\text{-}2)$. This settles the case $p \neq 2$ and $n = 7$.

Part (i) and (iii) of (b) now follow from the cases $n = 6, 7$ by induction, using Proposition 4. In order to prove Part (ii) of (b) we need an extra construction to handle the case $n = 8$, because the above construction for $n = 6$ worked only for $p \geq 7$.

We define (in characteristic 3) a tame cover $f_1 : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree 5 with ramification type $(2\text{-}2, 2\text{-}2, 5)$ and Galois group D_5. Since 3 does not divide the order of the Galois group, we may use Riemann's Existence Theorem. For example, we may choose f_1 such that it corresponds to the triple $((1, 2)(3, 4); (1, 5)(2, 3); (1, 5, 2, 4, 3))$.

We also define (in characteristic 3) a cover

$$f_2(x) \;=\; \frac{x^5(x+1)}{x-1} \;=\; \frac{(x^2-x-1)^2(x^2+1)}{x-1} + 1.$$

The cover $f_2 : Y_2 \to \mathbb{P}^1_k$ is branched at $x = 0, 1, \infty$, with ramification type $(5, 2\text{-}2, 5)$. One easily checks that this is the only such cover in characteristic 3, up to isomorphism. Let $G_2 \subset A_6$ be its monodromy group. One checks, for example using GAP, that G_2 is either isomorphic to $\mathrm{PSL}_2(5) \cong A_5$ or to A_6.

We claim that $G_2 \simeq \mathrm{PSL}_2(5)$. (One does not actually need this in what follows.) It follows from [15, Proposition 7.4.3] that there is a unique $\mathrm{PSL}_2(5)$-Galois cover $g : Y \to \mathbb{P}^1_{\bar{\mathbb{Q}}}$ branched at three points with ramification type $(5, 2\text{-}2, 5)$ over $\bar{\mathbb{Q}}$, up to isomorphism. In fact, this cover may be defined over $\mathbb{Q}$. It follows from the discussion in [2, §2.4] that g has good reduction to characteristic $p = 3$. Here one uses that p exactly divides the order of $\mathrm{PSL}_2(5)$. Since f_2 is the unique cover (up to isomorphism) of degree 6 with ramification type $(5, 2\text{-}2, 5)$ in characteristic 3, we conclude that the reduction of g to characteristic 3 is isomorphic to f_2. This shows that $G_2 \cong \mathrm{PSL}_2(5)$.

Let $g_1 \in G_2$ (resp. $g_3 \in G_2$) be canonical generators of inertia at the point $x = 0$ (resp. $x = \infty$), with respect to some fixed 5th root of unity. Let $f_3 = f_1$. For $i = 1, 3$, we choose an embedding of the monodromy group G_i of f_i into $\mathrm{PSL}_2(5)$ by identifying G_i with the normalizer in $\mathrm{PSL}_2(5)$ of the subgroup generated by g_i. We may choose this identification such that the canonical generator of inertia of the ramification point of order 5 of f_i is g_i^{-1}. Using the results of §2.1, we may patch the covers f_1, f_2, f_3, yielding a cover $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree 6 with monodromy group $G_2 \subset S_6$ and ramification type $(2\text{-}2, 2\text{-}2, 2\text{-}2, 2\text{-}2, 2\text{-}2)$.

We now apply the construction of Proposition 4 to f. Note that it does not matter for the construction that the monodromy group of f is $\mathrm{PSL}_2(5)$ and not A_6. This yields a cover $f' : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree 8 branched at 7 points of type $(2\text{-}2, \ldots, 2\text{-}2)$. Note that the monodromy group G' of f' is a transitive group on 8 letters which contains $\mathrm{PSL}_2(5)$ and is contained in A_8. One checks, e.g. using GAP, that the only such group is A_8 itself. This proves (b) for $p = 3$ and $n = 8$. Now Part (ii) of (b) follows from the cases $n = 7, 8$ by induction, using Proposition 4. This completes the proof of (b).

Let us now prove (c) and (d). We suppose that $p = 2$. The case $n = 3$ is again trivial; the general case of (c) follows by induction, using Proposition

4. To prove (d), we have to start the induction with $n = 6$. We define (in characteristic 2)

$$f(x) = \frac{x^3\,(x^3 + x^2 + 1)}{x + 1} = \frac{(x^2 + x + 1)^3}{x + 1} + 1. \tag{2}$$

One checks that f is a tame cover of degree 6, branched at $\{0, 1, \infty\}$ and with ramification type $(3, 3\text{-}3, 5)$. A standard result in group theory (see e.g. [10], p. 171) shows that the monodromy group of f is isomorphic to A_6. This settles the case $(p, n) = (2, 6)$. Part (d) of the proposition follows now from the cases $n = 6$ by induction, using Proposition 4. This completes the proof of Proposition 7. □

The statements of Proposition 7 is probably not optimal. In the next proposition we list all the cases where we can positively exclude the existence of a tame rational function on $\mathbb{P}^1$ with alternating monodromy.

Proposition 8 (i) *Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a tame rational function of degree $n = 3$ or $n = 4$, with monodromy group A_n. Then the ramification type of f is $(3, \ldots, 3)$ (with $n - 1$ branch points).*

(ii) *Suppose that $p = 2$. Then there does not exist a tame rational function $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree 4 and with monodromy group A_4.*

(iii) *Suppose that $p = 3$. Then there does not exist a tame rational function $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree $n \leq 5$ with monodromy group A_n.*

Proof: The proof of (i) is trivial. To prove (ii), suppose that there does exist a tame cover $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree 4 with monodromy group A_4. By (i), such a cover would be of type $(3, 3, 3)$. Such a cover would also lift to characteristic 0, in a unique way. It is easy to check, using the rigidity criterion ([15, Section 7.3]), that there is a unique A_4-cover of $\mathbb{P}^1$ with ramification type $(3, 3, 3)$ in characteristic zero. Therefore, the cover we are looking for would be the reduction of the cover (1) to characteristic 2. However, it is easy to check that the cover (1) has bad reduction to characteristic 2. This gives the desired contradiction and proves (ii).

To prove (iii), suppose that $p = 3$. The cases $n = 3, 4$ are already excluded by (i). It is easy to see, using the Riemann–Hurwitz formula, that a tame cover $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree 5 has ramification type $(5, 5)$, $(5, 2\text{-}2, 2\text{-}2)$ or $(2\text{-}2, 2\text{-}2, 2\text{-}2, 2\text{-}2)$. In the first case the monodromy group is cyclic of order 5, in the second case a dihedral group of order 10. So it remains to rule out the third case. Let (g_1, g_2, g_3, g_4) be a 4-tuple of double transpositions in A_5 with $g_1 g_2 g_3 g_4 = 1$. We claim that the elements $g_1, \ldots, g_4$ do not genererate A_5. To prove the claim, let $h := g_1 g_2$ and G the group generated by $g_1, \ldots, g_4$. If $h = 1$ then G is generated by two elements of order 2; it is then isomorphic to $\mathbb{Z}/2$, S_3 or D_5, but not to A_5. If h is a 3-cycle, then the subgroups $G_1 := \langle g_1, g_2 \rangle \subset A_5$ and $G_2 := \langle g_3, g_4 \rangle$ are both isomorphic to S_3 and contain the same subgroup of order 3. It follows easily that $G_1 = G_2$ and hence that $G \cong S_3$. Finally, if h is a

5-cycle, then G_1 and G_2 are dihedral groups of order 10 and contain the same cyclic group of order 5. As before, it follows that $G = G_1 = G_2 \cong D_5$. This proves the claim, and completes the proof of Proposition 8. $\Box$

Remark 9 The following cases are left open by Proposition 7 and Proposition 8:

(i) $p = 2$, $n \geq 6$ even, and ramification type $(3, \dots, 3)$,

(ii) $p = 3$, $n = 6$,

(iii) $p = 5$, $n \geq 6$ even, and ramification type $(2\text{-}2, \dots, 2\text{-}2)$.

3.2 The following theorem is a more precise version of Theorem 1 of the introduction.

Theorem 10 *Let $g \geq 0$ and $n \geq 3$. Let X be a generic curve of genus g, defined over an algebraically closed field k of characteristic $p \geq 0$. Then there exists a tame rational function $f : X \to \mathbb{P}^1_k$ of degree n with monodromy group A_n in each of the following cases.*

(a) If $p \neq 2, 3$ and $n \geq \max(g+3, 2g+1)$, we may choose f with $r = g+n-1$ branch points and ramification type $(3, \dots, 3)$.

(b) Suppose that one of the following holds:

(i) *$p \geq 7$ and $n \geq \max(6+g, 2g+1)$,*

(ii) *$p = 3$ and $n \geq \max(7, 6+g, 2g+1)$, or*

(iii) *$p = 5$, $n \geq \max(7+g, 2g+1)$ and $n+g$ is odd.*

Then we may choose f with $r = g+n-1$ branch points and ramification type $(2\text{-}2, \dots, 2\text{-}2)$.

(c) If $p = 2$, $n \geq \max(g+3, 2g+1)$ and $n+g$ is odd, then we may choose f with $r = g+n-1$ branch points and ramification type $(3, \dots, 3)$.

(d) If $p = 2$, $n \geq \max(g+6, 2g+3)$ and $n+g$ is even, we may choose f with $r = g+n-3$ branch points, and ramification type $(5, 3\text{-}3, 3, \dots, 3)$.

Proof: We start with two observations. First, to prove the theorem we may extend the base field k by an arbitrary algebraically closed field extension k'/k. Indeed, if we can show that the curve $X \otimes k'$ admits a rational function with certain properties, then a standard specialization argument yields the existence of a rational function on X with the same properties. Second, by the irreducibility of $\mathcal{M}_g$, a generic curve of a given genus is unique, up to isomorphism and extension of the base field. We may hence speak about *the* generic curve of genus g. These two observations will allow us to prove the theorem by induction on the pair (g, n), using Proposition 5.

We will discuss the induction procedure in detail for Part (a). For Part (b)–(d), we will only give the necessary modifications.

Suppose that $p \neq 2, 3$. For $g = 0$, the statement of Theorem 10 (a) is equal to the statement of Proposition 7 (a). Fix an integer $g' \geq 1$ and suppose that we have already proved Theorem 10 (a) for all pairs (g, n) with $g < g'$. We have to show that Theorem 10 (a) holds for all pairs (g', n') with $n' \geq \max(g'+3, 2g'+1)$. Write $g' = g + 1$ and $n' = n + 1$. Let X be the generic curve of genus g over k. By the induction hypothesis, there exists a tame rational function $f : X \to \mathbb{P}^1$ of degree n with alternating monodromy, $r = g + n - 1$ branch points and ramification type $(3, \dots, 3)$. We also have $n > 2g + 1$ and hence $r > 3g$. Therefore, by Proposition 5, the generic curve X' of genus g' admits a tame rational function $f' : X' \to \mathbb{P}^1$ of degree n', with monodromy group $A_{n'}$ and ramification type $(3, \dots, 3)$. In other words, Theorem 10 (a) holds for the pair (g', n'). This completes the proof of (a).

The proof of (b) is almost the same. Note that for $g = 0$ the statement of Theorem 10 (b) reduces again to the statement of Proposition 7 (b). The only problem occurs for $p = 3$ and the pair $(g', n') = (1, 7)$, because Proposition 7 (b) says nothing about the case $(g, n) = (0, 6)$ (see also Proposition 9 (ii)). However, in the proof of Proposition 7 (b) we did construct a tame rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 6, with 5 branch points and ramification type (2-2, …, 2-2). We showed that its monodromy group is isomorphic to $\mathrm{PSL}_2(5) \cong A_5$. If we apply the construction underlying the proof of Proposition 5 to this cover, then we obtain a tame rational function $f' : X' \to \mathbb{P}^1$ on the generic curve of genus one of degree 7, with ramification type (2-2, …, 2-2). Its monodromy group is a transitive subgroup of A_7 and contains a subgroup isomorphic to A_5. One checks, e.g. using GAP, that this group must be A_7. This settles the case $(p, g, n) = (3, 1, 7)$ and completes the proof of (b).

The proof of (c) and (d) is again similar. The only difference is that we have a different bound on n in (d). However, the induction procedure used in the proof of (a) and (b) goes through. □

3.3 Open problems in characteristic 2 and 3 As we already mentioned in the introduction, our main result (Theorem 10) is optimal for $p \neq 2, 3$ (in a certain sense). However, in characteristic 2 and 3 there remain a number of cases which we could not handle with the methods of the present paper. We leave these remaining cases as problems for the interested reader.

Problem 11 Let X be the generic curve of genus 1 in characteristic 2. Show that there exists a tame rational function $f : X \to \mathbb{P}^1$ of degree 5 with monodromy group A_5 and ramification type $(3, 3, 3, 3, 3)$.

Let $g : E \to \mathbb{P}^1$ be the tame cyclic Galois cover of degree 3 with three branch points. Note that, in characteristic 2, the curve E is the unique supersingular elliptic curve. (In characteristic 0, E is an elliptic curve with complex multiplication by $\mathbb{Z}[\zeta_3]$.) Applying the construction underlying the proof of Proposition 4, we obtain a tame cover $f : X \to \mathbb{P}^1$ of degree 5 with monodromy group A_5,

ramification type $(3,3,3,3,3)$ and with generic branch points. By Riemann-Hurwitz, X has genus 1. We believe but have not been able to show that X is generic. The problem is the following. By construction, the cover $f: X \to \mathbb{P}^1$ is the generic fibre of a family of covers over a two-dimensional base. On a sublocus of codimension one of the base, this family degenerates to the cover $g: E \to \mathbb{P}^1$. In particular, over this sublocus the top curve of the cover is isotrivial. But it is not clear whether X is isotrivial or generic.

Note that the characteristic 0 version of Problem 11 is solved in the paper [4], using the same construction as above. The proof uses the braid action to show the X is not isotrivial. In fact, the case $(g,n)=(1,5)$ is the only case where the braid action enters into the proof of the main result of [12] in an essential way.

Remark 12 At the moment, we do not know whether the generic curve of genus g in characteristic 2 admits a tame rational function of degree n with alternating monodromy, in each of the following cases:

(i) $(g,n)=(1,5)$,

(ii) $g \geq 3$ is odd and $n=2g+1$,

(iii) $g \geq 2$ is even and $n=2g+2$.

If Problem 11 (which corresponds to Case (i)) had a positive solution, then also Case (ii) and (iii) would be solved, as one can see from the induction procedure in the proof of Theorem 10. If this were the case, then we could omit the condition '$n+g$ is odd' in the statement of Theorem 10 (c) (but we would have to add the condition '$(g,n) \neq (0,4)$'). This would make Part (d) of Theorem 10 superfluous, and would give an 'optimal' result even in characteristic 2.

In characteristic 3, the existence of tame rational functions with alternating monodromy is open in the following (finite) list of cases:

$$(g,n) = (1,5),(1,6),(2,5),(2,6),(2,7),(3,7),(3,8),(4,9).$$

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